The Distribution of The Non – Stationary Queue Size of The System M | G | 1 | N.

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ABSTRACT

In the work is found Laplace's transformation for the distribution of non-stationary length of the system with finite queue. From this expression is defined formula for stationary queue length distribution.

Keywords: Queue systems, non-stationary queue size (length), stationary queue length (size), busy period.

Let us consider a single-server queue system $F_N -$ M|G|1|N which is characterized as follows: the interarrival times of customers are mutually independent random variables having the common distribution function

$$A(x) = \begin{cases} 1 - e^{-\lambda x}, & \lambda > 0, x > 0, \\ 0, & x \le 0. \end{cases}$$

And the service times are also mutually independent and identically distributed random variables with distribution function $B(x)$, $[B(+0) = 0]$ and

$$\mu^{-1} = \int_0^\infty x dB(x).$$

The customers are served in the order of their arrivals. Suppose that the number of customers in the system is at most N+1, that is, if an arriving customer finds N+1 customers (including the served customer), then he deports without being served.The service starts at time t = 0 in the presence in the system j (j> 0) customers one of which behave in the server. We suppose that the server is busy if there is at least one customer in the system.

In what follows super script in the bracket denotes the number of customers in the system at the moment t=0 including the served customer.

We shall denote by $\xi^j(t)$ the queue size, that is, the total number of customers in the system at time t $(0 \le t < \infty)$;

$\varsigma_1^{(j)} -$ is the initial busy period and $æ_1 -$ the initial empty period of the system. If j = 1we have an ordinary busy period.

It is clear that the random process is evolved in turn on the busy periods and empty state periods of the system. The process that is considered for $t \in (0, \varsigma_1^{(j)})$ we shall call conditionally process on the busy period.

We shall use the following notations:

$$P^{(j)}(k,t) = P\left(\xi^{(j)}(t) = k, \varsigma_1^{(j)} \ge t\right), \quad k = \overline{1, N+1},$$

$$g^{(j)}(t) = \frac{d}{dt} P\left(\varsigma_1^{(j)} < t\right),$$

$$\overline{P}^{(j)}(k,s) = \int_0^\infty e^{-st} P^{(j)}(k,t) dt, \quad k = \overline{1, N+1}, \quad Res \ge 0,$$

$$\hat{\overline{P}}^{(j)}(v,s)=\sum_{k=1}^{N+1}v^k\,\overline{P}^{(j)}(k,s)\,,$$

$$\overline{g}^{(j)}(s)=\int_0^\infty e^{-st}\,g^{(j)}(t)dt\,,\ \ Res\geq 0,$$

$$\overline{b}(s)=\int_0^\infty e^{-st}\,dB(t),\qquad Res\geq 0.$$

In the work [2] it is established the relation for $t\in(0,\varsigma_1^{(j)})$:

$$\hat{\overline{P}}^{(j)}(v,s)=\frac{v^{j+1}-v\,\overline{g}^{(j)}(s)+\lambda(1-v)v^{N+1}\overline{D}^{(j)}(v,s)}{v-\overline{b}(s+\lambda-\lambda v)}\cdot\frac{1-\overline{b}(s+\lambda-\lambda v)}{s+\lambda-\lambda v}\,,\qquad (1)$$

where $\overline{D}^{(j)}(v,s)=\overline{P}^{(j)}(N+1,s)Q(v,s)$, $|Q(v,s)|<\infty$ for $Res\geq 0$ and $|v|\leq 1$.

In the considered paper by using methods of the renewal theory we express characteristics of the process $\xi^j(t)$ for arbitrary $t\epsilon[0,\infty)$ by characteristics of this process obtained for $t\in(0,\varsigma_1^{(j)})$. This method was first offered by Gaver D.F. [3] and used by many other authors.

We introduce the notations

$$\Pi^{(j)}(k,t)=P\big(\xi^{(j)}(t)=k\big),\qquad j>0,\quad k=\overline{1,N+1}.$$

$$\overline{\Pi}^{(j)}(k,s)=\int_0^\infty e^{-st}\,\Pi^{(j)}(k,t)dt\,,\ \ Res\geq 0\,,$$

$$\hat{\overline{\Pi}}^{(j)}(v,s)=\sum_{k=0}^{N+1}v^k\,\overline{\Pi}^{(j)}(k,s)\,,\quad |v|\leq 1\,.$$

Theorem 1. For $Res\geq 0$ and $|v|\leq 1$ holds the following relation:

$$\hat{\overline{\Pi}}^{(j)}(v,s)=\frac{\overline{g}^{(j)}(s)}{s+\lambda-\lambda\overline{g}^{(1)}(s)}\cdot\left\{1+\lambda\hat{\overline{P}}^{(1)}(v,s)\right\}+\hat{\overline{P}}^{(j)}(v,s),\qquad (2).$$

Proof of the theorem 1. Between the probabilities $\Pi^{(j)}(k,t)$, $k=\overline{1,N+1}\,,j=\overline{1,N}$, and $P^{(j)}(k,t)$ $(k=\overline{1,N+1}\,,j=\overline{1,N})$ it is possible to establish the following equality:

$$\Pi^{(j)}(k,t)=P\big(\xi^{(j)}(t)=k\big)=P\left(\xi^{(j)}(t)=k,\varsigma_1^{(j)}\geq t\right)+P\left(\xi^{(j)}(t)=k,\varsigma_1^{(j)}<t\right).\qquad (3)$$

Since for $k\geq 1$

$$P\left(\xi^{(j)}(t)=k,\varsigma_1^{(j)}<t\right)=P\left(\xi^{(j)}(t)=k,\varsigma_1^{(j)}+æ_1<t\right),$$

the relation (3) takes the following form:

$$\Pi^{(j)}(k,t)=P^{(j)}(k,t)+P\left(\xi^{(j)}(t)=k,\varsigma_1^{(j)}+æ_1<t\right).\qquad (4)$$

Let $\varsigma_1^{(j)}$, $\varsigma_2,\varsigma_3,\ldots$and $æ_1,æ_2,\ldots$ –are the sequence of busy periods and empty state periods of the system, respectively. Then, it is clear that the sequence

$$\mathcal{E}_1=\varsigma_1^{(j)}+æ_1\,,\qquad \mathcal{E}_n=\varsigma_n^{(1)}+æ_n\,,\qquad n\geq 2\,,$$

Are the renewal process. Since the customers arriving to the system constitute a Poisson flow, $\mathcal{E}_1,\mathcal{E}_2,\ldots$are mutually independent random variables and

$$P(\mathcal{E}_2<x)=P(\mathcal{E}_3<x)=\cdots\ ,\ x\geq 0,$$

for j = 1

$$P(\mathcal{E}_1<x)=P(\mathcal{E}_2<x)=\cdots\ ,\ x\geq 0.$$

Since $\mathcal{E}_1=\varsigma_1^{(j)}+æ_1$is the renewal moment, then we obtain

$$P\left(\xi^{(j)}(t)=k,\ \varsigma_1^{(j)}+æ_1<t\right)=\int_0^t P\left(\xi^{(j)}(t)=k,u\leq\varsigma_1^{(j)}+æ_1<u+du\right)=$$

$$= \int_0^t P\left(\xi^{(j)}(t-u) = k\right) \cdot P(u \le \varsigma_1^{(j)} + æ_1 < u + du\right) =$$

$$= \int_0^t \Pi^{(1)}(k, t-u) dP\left(\varsigma_1^{(j)} + æ_1 < u\right).$$

Thus taking this into account from (4) we have

$$\Pi^{(j)}(k,t) = P^{(j)}(k,t) + \int_0^t \Pi^{(1)}(k, t-u) dP\left(\varsigma_1^{(j)} + æ_1 < u\right).$$

From here keeping in mind that $P(æ_1 \ge x) = e^{-\lambda x}$, and passing to the Laplace transform we have

$$\overline{\Pi}^{(j)}(k,s) = \overline{P}^{(j)}(k,s) + \frac{\lambda}{\lambda+s}\overline{g}^{(j)}(s)\overline{\Pi}^{(1)}(k,s). \qquad (5)$$

Let now k = 0. Then holds the following relation:

$$P\left(\xi^{(j)}(t) = 0\right) = P\left(\xi^{(j)}(t) = 0, \varsigma_1^{(j)} + æ_1 \ge t\right) + P(\xi^{(j)}(t) = 0, \varsigma_1^{(j)} + æ_1 < t\right) =$$

$$= \int_0^t P(æ_1 > t-u) dP\left(\varsigma_1^{(j)}(t) < u\right) + \int_0^t \Pi^{(1)}(0, t-u) dP\left(\varsigma_1^{(j)} + æ_1 < u\right).$$

From here, passing to Laplac transform we obtain

$$\overline{\Pi}^{(j)}(0,s) = \overline{g}^{(j)}(s)\frac{1}{\lambda+s} + \overline{g}^{(j)}(s) \cdot \frac{\lambda\overline{\Pi}^{(1)}(0,s)}{\lambda+s}. \qquad (6)$$

Now passing to generating function in the relations (5) and (6) we have

$$\hat{\overline{\Pi}}^{(j)}(v,s) = \hat{\overline{P}}^{(j)}(v,s) + \frac{\overline{g}^{(j)}(s)}{\lambda+s} \cdot \left[1 + \lambda\hat{\overline{\Pi}}^{(1)}(v,s)\right]. \qquad (7)$$

If we set j = 1, than from here we may define $\hat{\overline{\Pi}}^{(1)}(v,s)$, according to which from (7) is obtained the equality (2).

Theorem 2. For *Res≥0* and $j = \overline{1,N}$ holds the following equalities:

$$\overline{\Pi}^{(j)}(0,s) = \frac{\Delta_{N-j}}{(s+\lambda)\Delta_N - \lambda\Delta_{N-1}},$$

$$\overline{\Pi}^{(j)}(1,s) = (s+\boldsymbol{\lambda})\overline{\Pi}^{(j)}(0,s)f_1,$$

$$\overline{\Pi}^{(j)}(k,s) = \overline{\Pi}^{(j)}(0,s)\{(s+\lambda)f_k - \lambda f_{k-1}\}, \quad 2 \le k \le j,$$

$$\overline{\Pi}^{(j)}(k,s) = \overline{\Pi}^{(j)}(0,s)\{(s+\lambda)f_k - \lambda f_{k-1}\} - f_{k-j}, \quad j \le k \le N,$$

$$\overline{\Pi}^{(j)}(N+1,s) = \frac{1 - s\overline{\Pi}^{(j)}(0,s)}{s} - \overline{\Pi}^{(j)}(0,s)\{(s+\lambda)\varphi_N - \lambda\varphi_{N-1}\} - \varphi_{N-j},$$

where $\varphi_n = \sum_{k=1}^n f_k$, $f_k = f_k(s)$ and $\Delta_{\boldsymbol{k}} = \Delta_{\boldsymbol{k}}(\boldsymbol{s})$ are coefficients of v^k on expanding in powers of v the functions

$$f(v,s) = \frac{v}{s+\lambda-\lambda v} \cdot \frac{1 - \overline{b}(s+\lambda-\lambda v)}{\overline{b}(s+\lambda-\lambda v) - v}, \qquad (8)$$

$$\Delta(v,s) = \frac{v\overline{b}(s) - \overline{b}(s+\lambda-\lambda v)}{(1-v)(v - \overline{b}(s+\lambda-\lambda v))},$$

respectively.

Proof of the theorem 2. By using the relation (8) we shall write the function $\hat{\overline{P}}^{(j)}(v,s)$ given in (1) in the form

$$\sum_{k=1}^{N+1} v^k \overline{P}^{(j)}(k,s) = \left[\overline{g}^{(j)}(s) - v^j\right] f(v,s) + \lambda(1 - v)v^N \overline{D}^{(j)}(v,s) f(v,s) \qquad (10)$$

If the function $(1-v)v^N\overline{D}^{(j)}(v,s)f(v,s)$ is expanded in powers of v, then in this expantion involve only members with v power of which greater than N. Therefore, in order to define the $\overline{P}^{(j)}(k,s)$, $k=\overline{1,N}$, it is sufficient to consider of expantion of the function $\left[\overline{g}_N^{(j)}(s) - v^j\right] f(v,s)$.

By virtye of the condition the equality of power serieses from (10) we obtain that

$$\overline{P}^{(j)}(k,s) = \overline{g}_N^{(j)}(s) f_k \ , \quad 1 \le k \le j \ , \qquad (11)$$
$$\overline{P}^{(j)}(k,s) = \overline{g}_N^{(j)}(s) f_k - f_{k-j} \ , \ j < k \le N \ ,$$

where $f_k = f_k(s) -$ is the function defined in the relation (8).

The function $\overline{P}^{(j)}(N+1,s)$is obtained from the trivial equality

$$\hat{\overline{P}}^{(j)}(1,s) = \sum_{k=1}^{N+1} \overline{P}^{(j)}(k,s) = \frac{1-\overline{g}^{(j)}(s)}{s} \qquad (12)$$

Now by using the formula (1) we define the function $\overline{\Pi}^{(j)}(k,s)$, $k=\overline{0,N+1}$:

$$\overline{\Pi}^{(j)}(0,s) = \frac{\overline{g}^{(j)}(s)}{s+\lambda-\lambda\overline{g}^{(1)}(s)} \ , \qquad (13)$$

$$\overline{\Pi}^{(j)}(k,s) = \overline{P}^{(j)}(k,s) + \frac{\lambda\overline{g}^{(j)}(s)}{s+\lambda-\lambda\overline{g}^{(1)}(s)} \overline{P}^{(1)}(k,s) \ , \quad k = \overline{1,N+1}. \qquad (14)$$

In the work [4] is given the following formula:

$$\overline{g}^{(j)}(s) = \frac{\Delta_{N-j}}{\Delta_N} \ . \ \ j > 0 \ ,$$

where $\Delta_k = \Delta_k(s) -$ is the function defined in (9). Taking into account this and relations (11), (12) from (13) and (14) we obtain the statement of the Theorem 2.

Remark. In the work [1] was shown existence of the limit

$$P(k) = \lim_{t\to\infty} P(\xi^{(j)}(t) = k) \ , \qquad k = \overline{0,N+1},$$

and found formulas defining $P(k)$. From Theorem 2 we may obtain these formulas passing to the limit when $s \to 0$.

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